

THE PROJECTIVE THEORY OF ORTHOPOLES

By

SISTER MARY CORDIA KARL

of the

School Sisters of Notre Dame

A DISSERTATION SUBMITTED TO THE BOARD OF
UNIVERSITY STUDIES OF THE JOHNS HOPKINS
UNIVERSITY IN CONFORMITY WITH THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

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THE PROJECTIVE THEORY OF ORTHOPOLES

By SISTER MARY CORDIA KARL, College of Notre Dame of Maryland

Introduction

The term "orthopole of a line with respect to a given triangle" was first used by Neuberg to designate a point constructed as follows:¹ Perpendiculars are drawn from the vertices of the given triangle $A_1A_2A_3$ to the given line l , cutting l in B_1, B_2, B_3 , respectively. Then are drawn lines B_1C_1, B_2C_2, B_3C_3 , perpendicular respectively to A_2A_3, A_3A_1, A_1A_2 . These three perpendiculars are concurrent on a point M which is the orthopole of l with respect to the given triangle. Neuberg showed that while the line l gave rise to a unique orthopole M , yet M was equally well determined by each of two other lines. The three lines that gave rise to a common orthopole he designated as "associated" lines. He also proved that when the orthopole varied on a line, the three associated lines varied on a line parabola.

It is the purpose of this paper to study this (1, 1) correspondence between lines and line parabolas, to deduce therefrom the (1,3) correspondence between orthopoles and their associated lines, and then to prove by means of projective geometry some theorems concerning orthopoles. The theorems, with the exception of 1 and its Corollary, 16, 17, and the construction for finding the lines associated with a given one, have all been given before. They may be found in the works of Neuberg,² Cwojdzinski,³ Goormaghtigh,⁴ Ramler.⁵ In the proofs as given by these authors, the methods of synthetic and analytical geometry have been used, various types of coordinate systems being employed.

Since the (1,3) correspondence will be defined by means of a net of line conics, some preliminary theorems concerning such nets will be cited.⁶

Theorem A. In a net of line conics, the point pairs that are the centers of the pencils that constitute the degenerate conics of the net, lie on a cubic,⁷ which, in the case of a base line, degenerates into that line, l , and a residual conic, C^2 .

The following theorems refer to nets of line conics having a base line.

Theorem B. There is a (1,1) correspondence between the points on l and those on C^2 , each pair of corresponding points forming a degenerate conic. In

¹ *Sur une transformation des figures*, Nouv. Corr. Math., vol. 4, p. 379, 1878.

² *Die Verwandtschaft zwischen einer Geraden und ihrem Lotpunkt in Bezug auf ein Dreieck*, Archiv der Mathematik und Physik, vol. 3, p. 89, 1902.

³ *Der Lotpunkt, ein neuer merkwürdiger Punkt des Dreiecks*, Archiv der Mathematik und Physik, vol. 1, p. 175, 1901.

⁴ *The Orthopole*, Tohoku Mathematical Journal, 1927, p. 78.

⁵ *The orthopole loci of some one-parameter systems of lines referred to a fixed triangle*, American Mathematical Monthly, vol. 37, p. 130. March 1930.

⁶ Proofs of these theorems may be found in the original copy of the writer's dissertation on the subject in the Johns Hopkins University library.

⁷ Rey, *Geometry of Position* (1898), p. 231.

this correspondence, the points common to l and C^2 correspond doubly, i.e., to each considered as a point on l corresponds the other considered as a point on C^2 , and to each considered as a point on C^2 corresponds the other considered as a point on l .

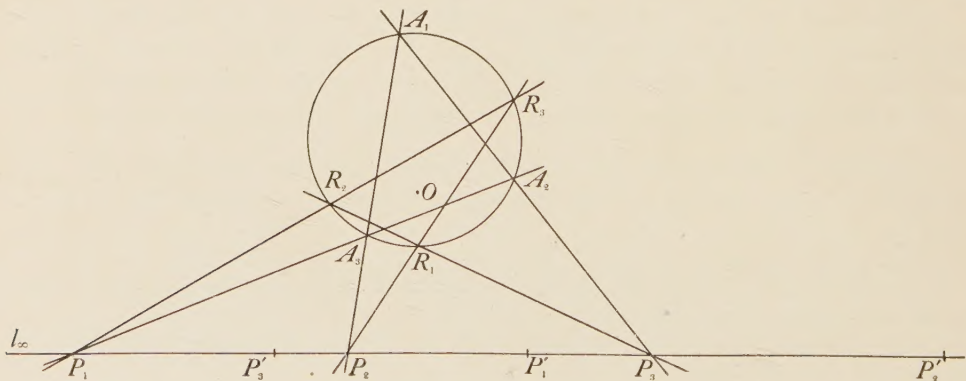
Theorem C. If the centers of the pencils that constitute degenerate conics of the net be joined to any point P on C^2 , these lines are conjugate pairs in an involution of lines on P .

Theorem D. If a line l and a conic C^2 be chosen arbitrarily, the net having these for the locus of the centers of the pencils that form the degenerate conics is fixed by assigning arbitrarily one point pair—one point on l , the other on C^2 —as one degenerate line conic of the net, the points common to l and C^2 being the centers of a second degenerate conic.

Theorem E. If 1, 2, 3 are three points on l which form with the three vertices of a triangle inscribed in C^2 three degenerate conics, and if $1', 2', 3'$ are the points in which the sides of the triangle opposite 1, 2, 3, respectively, cut l , then the three pairs $1, 1'; 2, 2'; 3, 3'$ are conjugate pairs of an involution.

Basis for Choice of Net and Determining Elements of the Correspondence

Neuberg proved that as the orthopole varies on the side A_1A_2 of the base triangle, the three associated lines vary on the degenerate line parabola having as centers of the two pencils of lines R_3 and P_3' ; on A_2A_3 , centers R_1 , P_1' ; on A_1A_3 , centers R_2 , P_2' ; where R_1, R_2, R_3 are points on the circumcircle diametrically opposite to A_1, A_2, A_3 ; and P_1', P_2', P_3' are conjugates in the absolute involution on l_∞ with respect to the points P_1, P_2, P_3 , these being the points of intersection of l_∞ and the sides of the triangle respectively opposite to A_1, A_2, A_3 .



The triangles $A_1A_2A_3$ and $R_1R_2R_3$, being perspective from center O of the circle, are also perspective from l_∞ . It is readily seen that the preceding is an example of Theorem E, the involution of points on l being the absolute involution.

Neuberg also proved that as a line varies on a point P on the circumcircle

of the base triangle, its orthopole varies on the Simson line of that point with respect to the base triangle. Since l_∞ cuts the circle in the circular points, I and J , its orthopole is the intersection of the Simson lines of I and J . These lines, however, coincide with l_∞ . Thus, any point on l_∞ is the orthopole of l_∞ . Also, the two lines associated with l_∞ pass one through I and the other through J , the two intersecting at some point K on the circle. It follows that as a point varies on l_∞ , the associated lines vary on the degenerate parabola IJ .

In selecting the elements that determine the (1,3) correspondence that will be defined later, the choice is so made that to the three sides of the base triangle $A_1A_2A_3$ will correspond the parabolas as given by Neuberg, and that to l_∞ will correspond IJ . Since four independent pairs of the (1,3) correspondence give rise to the same four pairs in the Neuberg definition, the two definitions must be equivalent.

Determination of the Net and the Correspondence

The net of line conics to be used is set up as follows:

(a) Let l_∞ be chosen as base line. Then all the conics are parabolas.
 (b) The absolute involution on l_∞ is fixed by its double points I and J . Let IJ be a degenerate conic of the net. Then, by Theorem B, C^2 must pass through I and J , and consequently is a circle.

(c) Let $R_1R_2R_3$ be any triangle inscribed in this circle, and let R_1R_2 , R_2R_3 , R_3R_1 cut l_∞ in P_3 , P_2 , P_1 , respectively. Also, let $P_1P'_1$, $P_2P'_2$, $P_3P'_3$ be conjugate pairs in the absolute involution. The last needed element to determine the net, according to Theorem D, is chosen as degenerate conic $R_1P'_1$. But then, by Theorem E, $R_2P'_2$, $R_3P'_3$ are also degenerate conics.

The base triangle $A_1A_2A_3$, with respect to which the orthopoles of lines of the plane are defined is chosen as the triangle perspective with $R_1R_2R_3$ from center O of the circle. The triangles will also be perspective from l_∞ .

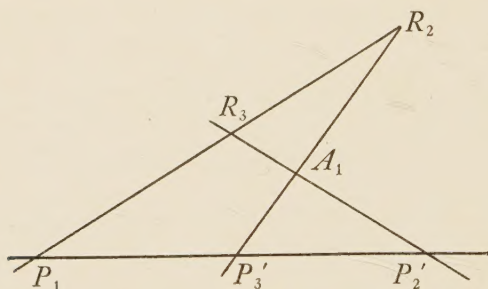
Consider now a projective correspondence between the lines of the plane and the ∞^2 line parabolas of the above linear system. In the (1,1) correspondence, to l_1 and l_2 , let correspond parabolas π_1 and π_2 , respectively. Then to the intersection of l_1 and l_2 must correspond the range determined by π_1 and π_2 . A (1,3) correspondence between points and lines may now be defined. To the point of intersection of l_1 and l_2 correspond the three lines, exclusive of l_∞ , common to all parabolas of the range determined by π_1 and π_2 . In order that the correspondence be that of orthopoles and associated lines, the determining pairs of homologous elements are chosen as follows: To A_1A_2 , let correspond the degenerate parabola $R_3P'_3$; to A_2A_3 , $R_1P'_1$; to A_1A_3 , $R_2P'_2$; and to l_∞ , IJ .

Further Correspondences Derived from the Defining Elements

Since A_1 is common to A_1A_2 and A_1A_3 , to A_1 corresponds the range containing degenerate parabolas $R_3P'_3$ and $R_2P'_2$. The three lines that correspond to A_1 are R_2R_3 , $R_2P'_3$, $R_3P'_2$, the two latter intersecting at A_1 on the circle. Also, R_2R_3 cuts l_∞ in P_1 . It follows that A_1P_1 is the third degenerate parabola of the range.

In terms of orthopoles, A_1 is the orthopole of R_2R_3 , $R_2P'_3$, $R_3P'_2$; and similarly, A_2 is orthopole of R_1R_3 , $R_1P'_3$, $R_3P'_1$; and A_3 , of R_1R_2 , $R_1P'_2$, $R_2P'_1$.

Since P_1 is common to A_2A_3 and l_∞ , to P_1 corresponds the range containing degenerate parabolas $R_1P'_1$ and IJ . Thus, P_1 is orthopole of R_1I , R_1J , and l_∞ ; i.e., l_∞ is a double tangent of parabolas of the range. If in the figure below, two adjacent sides of the quadrilateral become coincident, the points of contact of the inscribed parabolas with those two sides approach each other and also the point of intersection of the two sides. When the two sides coincide, the parabolas will be tangent at that point which forms with the remaining vertex a degenerate parabola. It follows that to P_1 corresponds the range containing degenerate parabolas $R_1P'_1$ and IJ , P'_1 being the point of contact; to P_2 , $R_2P'_2$ and IJ , P'_2 being the point of contact; and to P_3 , $R_3P'_3$ and IJ , P'_3 being the point of contact.



The point pairs that form the centers of the pencils which are the degenerate conics of the net lie one on the circle and the other on l_∞ . Also the three lines that correspond to a given point are always the sides, exclusive of l_∞ , of the quadrilateral in which parabolas of the range are inscribed. It follows that the three associated lines always intersect by pairs in three points on the circle. These sets, of three points each, constitute a linear series, a g_3^2 , on the circle. It is known that to every g_n^r on a plane algebraic curve, there corresponds a curve of order n , in a space S_r , in (1,1) correspondence with the given curve, on which the image of the g_n^r is the series of ∞^r points cut out by the hyperplanes of that space. Therefore the image of the g_3^2 on the circle is the series of ∞^2 points cut out on a plane cubic curve by the ∞^2 lines of the plane of that cubic, i.e., a g_3^2 on a plane cubic curve.

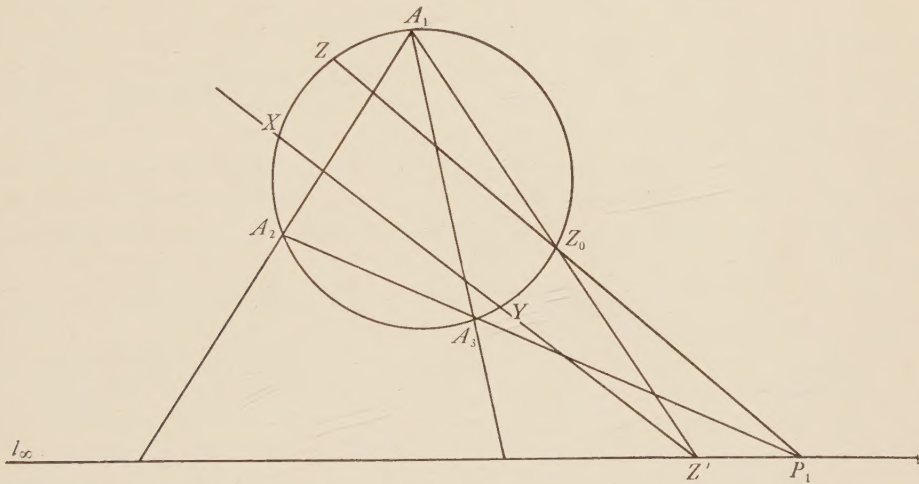
From this we deduce that there are three types of sets of points on the circle:

- (1) Three distinct points, and thus three distinct associated lines—corresponding to the three points in which a generic line cuts the cubic.
- (2) Two of the points coincident, and thus two of the associated lines coincident, the third tangent to the circle at the double point—corresponding to the points in which a tangent line cuts the cubic.
- (3) Sets having two points a neutral pair of the g_3^2 , in common, and a third

variable point—corresponding to the points in which a line through the double point of the cubic cuts it. It was shown previously that I and J belong to more than one set. They must therefore constitute this neutral pair. For sets of this type, the three associated lines are l_∞ and two others, one on I , the other on J , giving rise to ranges having only two degenerate conics, with a fixed point of contact on l_∞ .

Construction

Problem. Given the line XY and the base triangle $A_1A_2A_3$, to construct the two lines XZ and YZ , associated with XY .



Let XY cut l_∞ in Z' . Join A_1 to Z' , letting A_1Z' cut the circle again in Z_0 . Join Z_0 to P_1 , and let the join cut the circle at Z . Then ZX and ZY are the required lines.

Proof. Let π be the projectivity between points on the circle and points on l_∞ that together form degenerate conics of the net (Theorem B). Then

$$\pi(IJA_1A_2A_3R_1R_2R_3) = JIP_1P_2P_3P'_1P'_2P'_3.$$

Let K be any point on the circle. Then

$$KI, KJ, KA_1, \dots \bar{\wedge} KJ, KI, KP_1, \dots$$

Since this latter projectivity interchanges the pair KI, KJ , it is an involution. Now, for the pair KA_1, KP_1 , A_1P_1 is a degenerate conic. It follows from the involution, that the other points, M , in which KA_1 cuts l_∞ , and N , in which KP_1 cuts the circle must also constitute a degenerate conic. And finally, the line A_1N will cut l_∞ in the point L that forms with K a degenerate conic. There is thus an involution of points on the circle, center L .

In the construction as given, $K \equiv A_1$. Then $L \equiv P_1$. Let I_1 be the involution

of points on the circle, center P_1 ; and I_2 , the involution of lines on A_1 passing through conjugate pairs of I_1 on the circle. Then $I_1(Z) = Z_0$ and $I_2(A_1Z_0) = A_1Z$. Therefore, Z and Z' form a degenerate conic of the range inscribed in XY , XZ , YZ , and l_∞ . Thus, the lines XZ and YZ are associated with XY .

Some Theorems on Orthopoles

Theorem 1. Every point X on l_∞ is the orthopole of three lines, IJ , IK , JK , where K is the point on the circle that forms with the conjugate of X in the absolute involution a degenerate conic.

It was stated previously that to points on l_∞ correspond ranges having only two degenerate parabolas, one being IJ , the other consisting of a point K on the circle and a fixed point of contact, K' , on l_∞ , for all parabolas of the range. There is thus a (1,1) correspondence between the orthopole X and the point K on the circle. Let this correspondence be π . Let the correspondence between the point K on the circle and K' on l_∞ that forms with it a degenerate conic be σ . Finally, let the correspondence between X and K' be ω .

Now $\pi(X) = K$ and $\sigma(K) = K'$. But $\omega(X) = K'$. Therefore, $\pi\sigma = \omega$. It is known that $\pi(P_1P_2P_3) = R_1R_2R_3$ and $\sigma(R_1R_2R_3) = P'_1P'_2P'_3$. Thus $\omega(P_1P_2P_3) = P'_1P'_2P'_3$. Therefore, ω must coincide with the absolute involution and thus the theorem is proved.

Remark. Since π transforms three real points on l_∞ into three real points on the circle, the projectivity π must be real.

Corollary. Each of the circular points I and J is the orthopole of l_∞ taken twice and the tangent OJ and OI , respectively, at the other circular point, where O is the center of the circle.

Since $\pi\sigma = \omega$, $\pi = \omega\sigma^{-1}$. Now $\omega(IJ) = IJ$, and $\sigma^{-1}(IJ) = JI$. Therefore $\pi(IJ) = JI$. It follows that I is the orthopole of the lines determined by the set IJJ , while J is the orthopole of the lines determined by the set IIJ . The three associated lines are then l_∞ counted twice and the tangent to the circle at the double point of the set which in each case is the circular point other than the orthopole.

Theorem 2. The orthocenter of the base triangle is the orthopole of its three sides.

Since $\omega(P'_1P'_2P'_3) = P_1P_2P_3$ and $\sigma^{-1}(P_1P_2P_3) = A_1A_2A_3$, $\pi(P'_1P'_2P'_3) = A_1A_2A_3$. Thus P'_1 has corresponding to it the range having IJ and A_1P_1 as degenerate parabolas, P_1 being the fixed point of contact. Similarly, to P'_2 corresponds the range having IJ and A_2P_2 as degenerate parabolas, with P_2 the fixed point of contact. And finally, to P'_3 corresponds the range having IJ and A_3P_3 as degenerate parabolas, with P_3 as fixed point of contact.

Since the ranges that correspond to the points A_1 and P'_1 have in common the degenerate parabola A_1P_1 , that parabola must correspond to the line $A_1P'_1$. Similarly, parabolas A_2P_2 and A_3P_3 correspond respectively to lines $A_2P'_2$ and $A_3P'_3$. Furthermore, since the three lines $A_1P'_1$, $A_2P'_2$, and $A_3P'_3$ are concurrent in H , the orthocenter of the triangle $A_1A_2A_3$, to H must correspond the range

inscribed in $A_1A_2A_3$ and l_∞ . Or, in terms of orthopoles, H is the orthopole of the three sides of the triangle $A_1A_2A_3$.

Theorem 3. To every degenerate conic of the net corresponds the Simson line of a point on the circle with respect to the base triangle.

To every point P of the plane corresponds a range of parabolas. To the three degenerate parabolas of the range correspond three lines on P . As P varies in the plane, these lines envelope a curve of class three. It has already been shown that the three sides and altitudes of the base triangle, also l_∞ , have degenerate parabolas corresponding to them. Furthermore, by Theorem 1, Corollary, $\pi(IJ) = JI$. This means that for the set IIJ , having l_∞ as double line, J is the point of contact and hence a point of the envelope. Similarly, I is a point of the envelope.

The envelope of the Simson lines with respect to a given triangle of points on its circumcircle is known to be a three-cusped hypocycloid, having l_∞ as double tangent at the circular points. Each of the seven lines above mentioned is a Simson line with respect to the base triangle. Thus the Simson line envelope and the envelope first considered have in common six tangents, one double tangent and its two points of contact. Dually, there are two cubics having in common six points, one double point, and common tangents at that double point—making a total of twelve intersections. The two cubics, having in common more than nine points, must coincide. Dually, the two envelopes under consideration must be the same. Therefore, every line which has a degenerate parabola corresponding to it is a Simson line, and conversely.

Theorem 4. A Simson line is always perpendicular to the double line of the degenerate parabola to which it corresponds.

Let AB be the double line of the degenerate parabola that corresponds to the Simson line, s , A being on l_∞ , B on the circle. Let AA' be a conjugate pair in the absolute involution. By Theorem 1, to A' corresponds the range having IJ and AB as degenerate parabolas. Therefore, all parabolas of the range correspond to lines on A' . Then the Simson line, s , to which AB corresponds must pass through A' , and consequently is perpendicular to line AB .

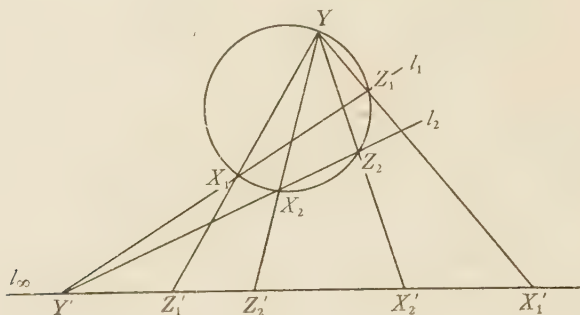
Theorem 5. The orthopole of an altitude of the base triangle is the foot of that altitude.¹

Theorem 6. When a line l moves parallel to itself, its orthopole moves on a line perpendicular to l , and the two associated lines pass through a fixed point on the circumference of the circle.

Let P be the point to which correspond the lines X_1Z_1 , YX_1 , YZ_1 . To the degenerate parabola YY' corresponds the Simson line perpendicular to the line YY' , and the Simson line must also pass through P . For any other position of the line on Y' , the range that corresponds to the new orthopole will always contain the degenerate parabola YY' . It follows that the orthopole of any line on Y' lies on the Simson line perpendicular to the line YY' . Also, the two lines

¹ Proofs of Theorems 5, 7, 9 may be found in original copy of writer's dissertation, loc. cit.

associated with the given line will always pass through that point on the circle that forms with Y' a degenerate parabola.



Theorem 7. The orthopole of a line perpendicular to a side of the base triangle is the foot of that perpendicular.

Theorem 8. As a line l varies on a curve of class m , its orthopole M varies on a curve of order $2m$.

As M moves on a line k , l envelopes a line parabola π . As M moves on a curve C , of order x , l envelopes a curve S of class m . To the common points of k and C , will correspond the lines common to π and S . These are $2m$ in number. Therefore the order x of C is $2m$.

Corollary. As a line m varies in a pencil of lines on a point P , the orthopole M varies on a conic, an ellipse.

In this case, m equals one; the locus of M is thus a conic. It remains to show that this is an ellipse, i.e., cuts l_∞ in two imaginary points. Since P is not on l_∞ , the only lines on P that can have orthopoles on l_∞ are PI and PJ . For P real, the other point of intersection, Q , of PI with the circle must be imaginary. The projectivity π between l_∞ and points on the circle, by the Remark, Theorem 1, is real. Therefore, the point R that forms a degenerate conic with Q is imaginary. To the range QIJ , then, corresponds the point R' , the conjugate of R in the absolute involution. But R' , also imaginary, is then the orthopole of QI or PI . Similarly, the orthopole of PJ is imaginary. Thus the conic which is the orthopole locus is an ellipse.

Theorem 9. When a line m varies in a pencil of lines on a point P on l_∞ , the orthopole locus degenerates into the Simson line perpendicular to m , and the line l_∞ .

Theorem 10. When a line varies on a point P on the circumcircle of the base triangle, the orthopole locus is the Simson line of the point P with respect to the base triangle taken twice.

The feet of the three perpendiculars from P to the sides of the base triangle are their respective orthopoles (Theorem 7). But these points determine the so-called Simson line of the point P with respect to the base triangle. By Theorem 6, there are always two lines on P that give rise to the same orthopole. Thus the Simson line must be traced out twice.

Corollary. The orthopole of a line cutting the circumcircle in points P and Q is the intersection of the Simson lines of P and Q .

Theorem 11. As a point M describes a curve of order n , the three associated lines of which M is orthopole envelope a curve of class $2n$.

As a line l varies on a point P , its orthopole describes a conic (Theorem 8, Corollary). This conic cuts the locus C of M in $2n$ points. To each of these points must correspond a line on P tangent to the envelope described by the lines as M varies on C . Therefore the class of the envelope is $2n$.

Theorem 12. As a line varies on the center O of the circumcircle of the base triangle, its orthopole varies on the nine-point circle of the base triangle.

The locus, a conic, must be a circle since it passes through the circular points I and J , the respective orthopoles of tangents OJ and OI (Theorem 1, Corollary). Also the feet of the three perpendiculars from O to the three sides of the base triangle are on the circle. Thus the required locus is the nine-point circle of the base triangle.

Theorem 13. As a line varies on the hypocycloid which is the Simson line envelope, the orthopole locus is a quartic, having a triple point at the orthocenter H , and passing through the feet of the three altitudes of the base triangle.

The hypocycloid being of class three, the orthopole locus must be of order six. This degenerates into l_∞ counted twice and a residual quartic. That l_∞ is a double line of the locus follows from the fact that l_∞ is a double tangent to the hypocycloid, and has as orthopole any point on l_∞ . Since the three sides of the base triangle are tangent to the hypocycloid, and each has H as orthopole, H is a triple point of the quartic. Finally, the altitudes are Simson lines having their respective feet as orthopoles. Thus the quartic also passes through the feet of the altitudes.

Theorem 14. As a line m varies as tangent to the circumcircle of the base triangle, its orthopole varies on the hypocycloid envelope of the Simson lines.

The tangents to the hypocycloid are all Simson lines and thus each gives rise to a degenerate conic of the net. From points not on the hypocycloid, three distinct tangents may be drawn. The corresponding range will thus have three distinct degenerate parabolas, and the associated lines will be distinct. But for points on the hypocycloid, two of the tangents will be coincident and the third distinct. The corresponding ranges will have two coincident degenerate parabolas, and the set of points on the circle has a double point. Of the three associated lines, one will be tangent to the circle at that double point, and the other will be the line joining the double point to the remaining point, this line being counted twice. Thus, for all points on the hypocycloid, one of the three associated lines will be tangent to the circle. Therefore, as a line varies as tangent to the circumcircle, its orthopole varies on the hypocycloid.

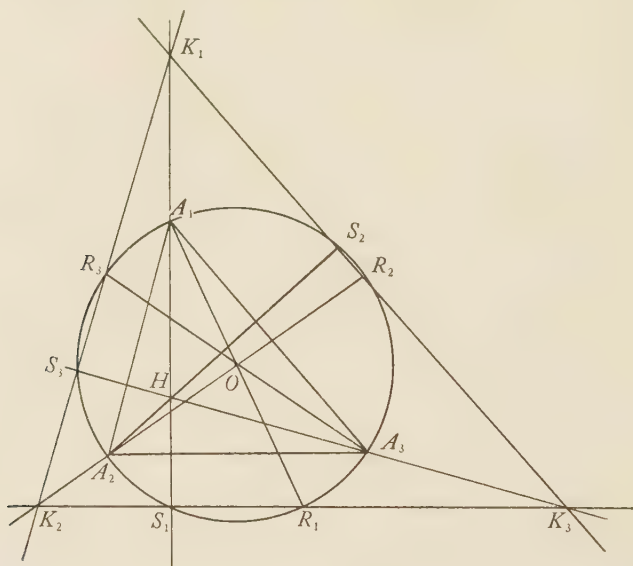
Corollary. If a line is tangent to the circumcircle, its orthopole is the point in which the Simson line of the point of contact touches the hypocycloid.

Theorem 15. Given any two sets of associated lines, the orthopole of either set with respect to the other as base triangle is the midpoint of the join of their orthocenters.

The following is a known property¹ of Simson lines: If the Simson lines of the points B_1, B_2, B_3 , with respect to the triangle $A_1A_2A_3$ are concurrent on a point M , then the Simson lines of A_1, A_2, A_3 , with respect to triangle $B_1B_2B_3$ are also concurrent on M , and M is midway between the orthocenters of the two triangles. From this property and the Corollary to Theorem 10, the above theorem obviously follows.

Theorem 16. As a point varies on the nine-point circle of the base triangle, its ortholines envelope a curve of class four, which degenerates into a three-cusped hypocycloid and the center of the circumcircle of the base triangle.

As the orthopole varies on a conic, the associated lines envelope a curve of class four. By Theorem 12, the center of the circumcircle is part of the desired locus. Consequently the residual portion is of class three.



Since one of the three associated lines is always on the center O , and thus a diameter, the other two lines—tangent to the envelope—intersect at right angles. Thus the circle is the orthoptic locus of the envelope. It is known that the orthoptic locus is a circle only for central conics and for curves of class three, order four, having l_∞ as double tangent at the circular points.² The above envelope must, therefore, be of order four, l_∞ being associated with diameters through both I and J , and touching the envelope at J and I . For all other diameters, the two associated lines cannot possibly coincide. The curve, then, has no other double tangents. Using the Plücker equations:

$$n = m(m - 1) - 2i - 3j$$

$$m = n(n - 1) - 2d - 3k$$

¹ Johnson, *Modern Geometry*, p. 211.

² Hilton, *Plane Algebraic Curves*, pp. 169-174.

and

$$i = 3n(n - 2) - 6d - 8k$$

for $n=4$, $m=3$, and $t=0$, it is found that $i=0$, $d=0$, and $k=3$. Therefore, the envelope is a three-cusped hypocycloid.

It will now be shown which hypocycloid this is. By the method indicated on page 331, the lines associated with A_1R_1 are found to be A_1S_1 and R_1S_1 ; with A_2R_2 , A_2S_2 and R_2S_2 ; with A_3R_3 , A_3S_3 and R_3S_3 .

It can readily be proven that the lines R_3S_3 , A_1S_1 , R_2S_2 are concurrent on K_1 ; R_2S_2 , A_3S_3 , R_1S_1 on K_3 ; and R_1S_1 , A_2S_2 , R_3S_3 , on K_2 . Therefore, the required hypocycloid must be tangent to the three sides of the triangle $K_1K_2K_3$, to its three altitudes, and to l_∞ twice at I and J . Then this hypocycloid must coincide with the Simson line envelope of points on the circumcircle of $K_1K_2K_3$ with respect to that triangle.

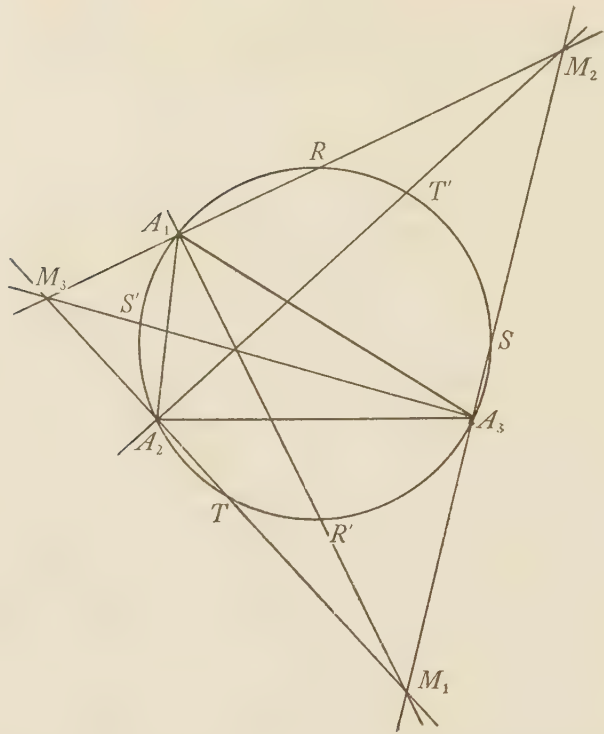
Theorem 17. As a point varies on the hypocycloid envelope of the Simson lines of the base triangle, the ortholines envelope the circumcircle of the base triangle and another three-cusped hypocycloid counted twice.

As the orthopole varies on a curve of order four, the ortholines have as envelope a curve of class eight. By Theorem 14, the circumcircle is part of this envelope. It will now be shown that the residual portion is a three-cusped hypocycloid, counted twice.

As a line moves parallel to itself, the two lines associated with it always pass through a fixed point on the circle. Also, the two lines associated with a tangent are always coincident. Now in every set of parallel lines, there will be two tangents to the circle at points T_1 and T_2 , say. Then PT_1 and PT_2 , each counted twice, will be associated with the tangents at the respective points, T_1 and T_2 . Furthermore, PT_1 and PT_2 are perpendicular.

Since one of the three lines that has a point on the Simson line envelope as orthopole is always tangent to the circle, the other two will always be a certain line counted twice. Also, since l_∞ is associated with the tangents to the circle at both I and J , l_∞ is a double tangent. In a given direction, then, there can be drawn only one tangent. And, from what was said above, any two tangents that are perpendicular must intersect on the circumcircle of the base triangle. The envelope consequently is a curve of class three, counted twice, having the circumcircle as orthoptic locus. The line at infinity being the only double tangent, it follows, as in Theorem 16, that the curve is a hypocycloid of three cusps.

The hypocycloid is determined as follows. The lines associated with a line parallel to any side of the base triangle pass through the opposite vertex. Let a and a' be tangents parallel to A_2A_3 ; b and b' , to A_1A_2 ; and c and c' , to A_1A_3 . Let the respective points of contact be R , R' ; S , S' ; T , T' . Then each of the following lines is tangent to the required hypocycloid: A_1R , A_1R' , A_2T , A_2T' , A_3S , A_3S' . It may readily be proven that A_1R , A_3S' , and A_2T are concurrent on M_3 ; A_1R , A_2T' , and A_3S , on M_2 ; and A_1R' , A_2T , and A_3S , on M_1 . The required



hypocycloid must coincide with the Simson line envelope of points on the circumcircle of triangle $M_1M_2M_3$ with respect to that triangle, each of the six lines mentioned above being such Simson lines.

BIOGRAPHY

Sister Mary Cordia Karl was born in New York City, November 16, 1893. After graduating from the parochial school, she attended Hunter High School and Hunter College, receiving from the latter institution the degree of Bachelor of Arts in 1916. She taught for several years in the New York City schools. In 1918 she entered the Order of the School Sisters of Notre Dame and has since been teaching at Notre Dame College, Baltimore, Md. In 1923 she enrolled at the Johns Hopkins University as a graduate student in the department of Mathematics and received the degree of Master of Arts in 1927. She again resumed work in the Mathematics Department in 1929.

